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# Superconformal Coset Equivalence from Level-Rank Duality

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We construct a one-to-one map between the primary fields of the  $N = 2$  superconformal Kazama-Suzuki models  $G(m, n, k)$  and  $G(k, n, m)$  based on complex Grassmannian cosets, using level-rank duality of Wess-Zumino-Witten models. We then show that conformal weights, superconformal  $U(1)$  charges, modular transformation matrices, and fusion rules are preserved under this map, providing strong evidence for the equivalence of these coset models.

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## 1. Introduction

One of the largest classes of non-trivial but still solvable two-dimensional conformally invariant field theories arises from the coset construction [1,2]. In this construction, the conformal algebra associated with the level  $k$  Kac-Moody algebra of some group  $G$  is orthogonally decomposed into the conformal algebra associated with some subgroup  $H$  and the  $G/H$  coset conformal algebra. In a similar way, superconformal coset algebras can be obtained from the orthogonal decomposition of the super Kac-Moody algebra associated with  $G$ . Kazama and Suzuki [3] showed that the superconformal algebra based on the coset  $G/H$  possesses an extended ( $N = 2$ ) superconformal symmetry if, for rank  $G = \text{rank } H$ , the coset  $G/H$  is a Kähler manifold. To establish that this algebra leads to a well-defined modular invariant conformal field theory requires consideration of the spectrum of primary fields, which involves issues of selection rules, field identifications [4,5,6], and fixed-point resolutions [7,8,9]. In this paper, we focus on the class of Kazama-Suzuki models based on the complex Grassmannian manifold  $\text{SU}(m+n)/[\text{SU}(m) \times \text{SU}(n) \times \text{U}(1)]$ . These superconformal coset models may be written as ordinary coset models

$$G(m, n, k) = \frac{\text{SU}(m+n)_k \times \text{SO}(2mn)_1}{\text{SU}(m)_{n+k} \times \text{SU}(n)_{m+k} \times \text{U}(1)_{mn(m+n)(m+n+k)}}, \quad (1.1)$$

where the  $\text{SO}(2mn)_1$  factor arises from the adjoint fermions of the super Kac-Moody algebra.

The invariance of the central charge of the coset models (1.1)

$$c^{m,n,k} = \frac{3mnk}{m+n+k} \quad (1.2)$$

under any permutation of  $m$ ,  $n$ , and  $k$  suggests that the models themselves may be invariant [3]. The invariance of the coset models under  $m \leftrightarrow n$  is manifest from their definition, but the further symmetry under  $k \leftrightarrow m$  is unexpected, as  $k$  and  $m$  play rather different roles. Kazama and Suzuki [3] showed that the supercurrent also respects the  $k \leftrightarrow m$  symmetry, providing further evidence for the conjecture. Gepner [10] demonstrated that the Landau-Ginzburg models corresponding to  $G(m, 1, k)$  and to  $G(k, 1, m)$  are equivalent, and Lerche *et. al.* [6] showed that the Poincaré polynomials of  $G(m, n, 1)$  and  $G(1, n, m)$  are identical. See also ref. [11].

In this paper, we construct an explicit one-to-one map between the primary fields of  $G(m, n, k)$  and  $G(k, n, m)$  when  $m$ ,  $n$ , and  $k$  have no common divisor, or only a prime

common divisor. When  $m$ ,  $n$ , and  $k$  have greatest common divisor  $p > 1$ , the model has fixed points that must be resolved into a multiplicity of fields to maintain modular invariance. Schellekens [8] has shown how to do this for  $p$  prime, and for this case we exhibit the one-to-one map between the resolved primary fields of  $G(m, n, k)$  and  $G(k, n, m)$ . We then demonstrate that the modular transformation matrices  $S$  and  $T$  are identical in the two theories. This further implies the equality (modulo integers) of conformal weights of corresponding primary fields, and the equality of the fusion rules via Verlinde's formula [12]. These identifications provide nearly conclusive evidence that  $G(m, n, k)$  and  $G(k, n, m)$  are equivalent conformal field theories. This equivalence arises largely as a consequence of the level-rank duality [13,14,15,16,17] of the constituent WZW models.

Level-rank duality has been shown to underlie equivalences between other coset models. Altschüler [18] has shown the equivalence of the conformal generators of various pairs of dual (non-superconformal) coset models, and the equivalence of the characters of certain non-unitary coset models was shown [19] to follow from the duality of principally-specialized characters [20]. Most closely related to the present work is that of Fuchs and Schweigert [21], who used the level-rank duality of orthogonal and symplectic groups [16] to show the equivalence of several pairs of  $N = 2$  superconformal models. In particular, they demonstrated the equivalence of the Kazama-Suzuki models

$$\frac{\mathrm{SO}(m+2)_k \times \mathrm{SO}(2m)_1}{\mathrm{SO}(m)_{k+2} \times \mathrm{U}(1)_{4(m+k)}} = \frac{\mathrm{SO}(k+2)_m \times \mathrm{SO}(2k)_1}{\mathrm{SO}(k)_{m+2} \times \mathrm{U}(1)_{4(m+k)}} \quad (1.3)$$

for  $m$  and  $k$  odd, and for  $m$  even and  $k$  odd but with a non-diagonal modular invariant in the theory on the right hand side. They also showed an isomorphism between several other sets of coset models based on non-hermitian symmetric spaces [22].

This paper is organized as follows: in section 2, we describe the Kazama-Suzuki model  $G(m, n, k)$  in some detail. Section 3 reviews level-rank duality between  $\mathrm{SU}(N)_K$  and  $\mathrm{SU}(K)_N$ . In section 4, we construct the map between primary fields of  $G(m, n, k)$  and  $G(k, n, m)$ , and demonstrate that the conformal weights and modular transformation matrices of corresponding fields are the same. Section 5 describes the map between the chiral rings of  $G(m, 1, k)$  and  $G(k, 1, m)$ . In section 6, we discuss the fixed-point resolution when  $m$ ,  $n$ , and  $k$  possess a common (prime) divisor, and construct the one-to-one map between resolved primary fields. Some concluding remarks form section 7.

## 2. Primary fields of the complex Grassmannian coset model

In this section, we describe the relevant details of the complex Grassmannian Kazama-Suzuki model (1.1). The central charge of this model (1.2) is obtained from

$$c^{m,n,k} = c^{m+n,k} - c^{m,n+k} - c^{n,m+k} + mn - 1 \quad (2.1)$$

where  $c^{N,K} = K(N^2-1)/(K+N)$  denotes the central charge of the  $SU(N)_K$  WZW model, and  $mn$  and  $1$  are the central charges of the  $SO(2mn)_1$  and  $U(1)$  models respectively.

The characters of the coset model in the absence of fixed-point subtleties<sup>1</sup> are given by the branching functions  $b_{\lambda_1, \lambda_2, q}^{\lambda_0, \pi}(\tau)$  in the character decomposition

$$\chi^{\lambda_0, \pi}(\tau) = \sum b_{\lambda_1, \lambda_2, q}^{\lambda_0, \pi}(\tau) \chi^{\lambda_1, \lambda_2, q}(\tau) \quad (2.2)$$

where  $\lambda_0$ ,  $\pi$ ,  $\lambda_1$ ,  $\lambda_2$ , and  $q$  denote primary fields of  $SU(m+n)_k$ ,  $SO(2mn)_1$ ,  $SU(m)_{n+k}$ ,  $SU(n)_{m+k}$ , and  $U(1)_{mn(m+n)(m+n+k)}$  respectively. Thus, primary fields of the coset model are labelled by the multi-index

$$\Lambda = \begin{pmatrix} \lambda_0 & \pi \\ \lambda_1 & \lambda_2 & q \end{pmatrix} \quad (2.3)$$

subject to the selection rules and identifications specified below.

Primary fields of the  $SU(N)_K$  WZW model are labelled by integrable representations  $\lambda$  of the  $SU(N)_K$  Kac-Moody algebra, those with non-negative extended Dynkin indices  $\{a_0, a_1, \dots, a_{N-1}\}$ , where  $a_0 = K - \sum_{i=1}^{N-1} a_i$ . These representations have Young tableaux whose first row length  $\ell_1 = \sum_{i=1}^{N-1} a_i$  is no greater than  $K$  boxes. Primary fields of  $SO(2N)_K$  are labelled by integrable representations  $\pi$  of  $SO(2N)_K$ , again with non-negative extended Dynkin indices  $\{a_0, a_1, \dots, a_N\}$ , where now  $a_0 = K - a_1 - 2 \sum_{i=2}^{N-2} a_i - a_{N-1} - a_N$ . For  $SO(2N)_1$ , there are only four: the singlet (1), vector ( $v$ ), spinor ( $s$ ), and conjugate spinor ( $c$ ) representations, which correspond to non-zero  $a_0$ ,  $a_1$ ,  $a_{N-1}$ , and  $a_N$  respectively. In the following, we will refer to the first two of these as the Neveu-Schwarz (NS) sector, and the last two as the Ramond (R) sector, alluding to the fermionic origin of the  $SO(2mn)_1$  factor.  $U(1)_L$  has  $L$  primary fields, labelled by the integers  $q = 0, 1, \dots, L-1 \bmod L$ .

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<sup>1</sup> We will deal with fixed points in section 6.

## Conformal weights and modular transformation matrices

The modular transformation matrices of the coset model characters

$$\begin{aligned}\chi_\Lambda(-1/\tau) &= \sum_{\Lambda'} S_{\Lambda\Lambda'} \chi_{\Lambda'}(\tau) \\ \chi_\Lambda(\tau + 1) &= T_{\Lambda\Lambda} \chi_\Lambda(\tau) = e^{2\pi i(h_\Lambda - c/24)} \chi_\Lambda(\tau)\end{aligned}\tag{2.4}$$

can be inferred from those of the branching functions (2.2). When there are no fixed points<sup>2</sup>, the coset modular matrix  $S$  is

$$S_{\Lambda\Lambda'}^{m,n,k} = mn(m+n) S_{\lambda_0\lambda'_0} S_{\lambda_1\lambda'_1}^* S_{\lambda_2\lambda'_2}^* S_{\pi\pi'} S_{qq'}^* \tag{2.5}$$

where  $S_{\lambda\lambda'}$  are the modular transformation matrices of the  $SU(N)_K$  WZW models [23],

$$S_{\pi\pi'} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-N} & -i^{-N} \\ 1 & -1 & -i^{-N} & i^{-N} \end{pmatrix} \quad \text{for } SO(2N)_1, \tag{2.6}$$

and

$$S_{qq'} = \frac{1}{\sqrt{L}} \exp\left(-\frac{2\pi iq q'}{L}\right) \quad \text{for } U(1)_L. \tag{2.7}$$

The factor of  $mn(m+n)$  in eq. (2.5) results from field identification, discussed below. Similarly, eqs. (2.2) and (2.4) determine the conformal weights of the coset primary fields modulo integers

$$h_\Lambda^{m,n,k} = h_{\lambda_0}^{m+n,k} - h_{\lambda_1}^{m,n+k} - h_{\lambda_2}^{n,m+k} + h_\pi^{2mn} - h_q \mod \mathbf{Z} \tag{2.8}$$

where

$$\begin{aligned}h_\lambda^{N,K} &= \frac{C_2(\lambda)}{K+N} && \text{for } SU(N)_K \\ h_\pi^{2N} &= (0, \frac{1}{2}, \frac{1}{8}N, \frac{1}{8}N) \quad \text{if } \pi = (1, v, s, c) && \text{for } SO(2N)_1 \\ h_q &= \frac{q^2}{2L} \mod \mathbf{Z} && \text{for } U(1)_L\end{aligned}\tag{2.9}$$

with  $C_2(\lambda)$  the quadratic Casimir of the representation  $\lambda \in SU(N)$ , here normalized to  $C_2(\text{adjoint}) = N$ .

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<sup>2</sup> See section 6 for modifications when fixed points are present.

## Superconformal U(1) charges

The  $N = 2$  superconformal symmetry of  $G(m, n, k)$  implies that the primary field  $\Lambda$  carries a superconformal U(1) charge  $Q_\Lambda$  given by [3]

$$Q_\Lambda = f_{2mn}(\pi) - \frac{q}{m+n+k} \pmod{2} \quad (2.10)$$

where

$$f_{2N}(\pi) = (0, 1, \frac{1}{2}N, \frac{1}{2}N - 1) \quad \text{for } \pi = (1, v, s, c) \quad \text{of } \text{SO}(2N)_1. \quad (2.11)$$

## Selection Rules

The branching function  $b_{\lambda_1, \lambda_2, q}^{\lambda_0, \pi}(\tau)$  will be non-vanishing if and only if  $(\lambda_1, \lambda_2, q)$  is contained in  $(\lambda_0, \pi)$  or its descendants. This translates into a constraint on the conjugacy classes of the representations, which is embodied in the following two selection rules [5,6]:

$$\begin{aligned} q &= -mr_{\lambda_0} + (m+n)r_{\lambda_1} + m(m+n)(a + \frac{1}{2}n\epsilon) \pmod{mn(m+n)(m+n+k)} \\ q &= nr_{\lambda_0} - (m+n)r_{\lambda_2} + n(m+n)(b + \frac{1}{2}m\epsilon) \pmod{mn(m+n)(m+n+k)} \end{aligned} \quad (2.12)$$

where  $r_\lambda$  denotes the number of boxes in the Young tableau corresponding to  $\lambda \in \text{SU}(N)_K$  (equivalently,  $r_\lambda = \sum_{i=1}^{N-1} ia_i$ ),  $a$  and  $b$  are integers defined modulo  $n(m+n+k)$  and  $m(m+n+k)$  respectively, and  $\epsilon = 0$  or  $1$  if  $\pi$  belongs to the NS or R sector respectively. Eliminating  $q$  between these equations yields a constraint between  $a$  and  $b$ :

$$ma - nb = r_{\lambda_0} - r_{\lambda_1} - r_{\lambda_2} \pmod{mn(m+n+k)}. \quad (2.13)$$

If  $(m, n)$ , the greatest common divisor of  $m$  and  $n$ , is greater than 1, then eq. (2.13) also constrains which representations  $\lambda_0$ ,  $\lambda_1$ , and  $\lambda_2$  are allowed:  $r_{\lambda_0} - r_{\lambda_1} - r_{\lambda_2}$  must be a multiple of  $(m, n)$ .

## Field identification

Not all  $\Lambda$  correspond to distinct primary fields of the coset model. Consider the two operations [5,6]:

$$\begin{aligned} J_1(\Lambda) &= \begin{pmatrix} \sigma(\lambda_0) & \sigma^n(\pi) \\ \sigma(\lambda_1) & \lambda_2 & q + n(m+n+k) \end{pmatrix}, \\ J_2(\Lambda) &= \begin{pmatrix} \sigma(\lambda_0) & \sigma^m(\pi) \\ \lambda_1 & \sigma(\lambda_2) & q - m(m+n+k) \end{pmatrix}, \end{aligned} \quad (2.14)$$

in which  $\sigma$  is related to a symmetry of the extended Dynkin diagram of the Kac-Moody algebra, and is defined as follows:

Acting on an  $SU(N)_K$  representation  $\lambda$ , the operation  $\sigma$  rotates the extended Dynkin indices:  $a_i(\sigma(\lambda)) = a_{i-1}(\lambda)$ , where  $a_{i+N} \equiv a_i$ . Alternatively,  $\sigma(\lambda)$  results from tensoring  $\lambda$  with the representation whose Young tableau consists of a single row of width  $K$  (a “cominimal” representation [24] or simple current [25]). Hence, the tableau representing  $\sigma(\lambda)$  is obtained from that of  $\lambda$  by adding a single row of  $K$  boxes to the top, and  $r_{\sigma(\lambda)} = r_\lambda + K \bmod N$ . We call  $\lambda$  and  $\sigma(\lambda)$  “cominimally equivalent,” and the set of representations  $\lambda, \sigma(\lambda), \dots, \sigma^{N-1}(\lambda)$  constitute a cominimal equivalence class [16], or simple current orbit [25].

Acting on the  $SO(2N)_1$  representation  $\pi$ , the operation  $\sigma$  exchanges the extended Dynkin indices  $a_0 \leftrightarrow a_1$  and  $a_{N-1} \leftrightarrow a_N$ . (This is the product of the operations  $\sigma$  and  $\varepsilon$  defined in ref. [16]). Hence  $\sigma(1) = v$ ,  $\sigma(v) = 1$ ,  $\sigma(s) = c$ , and  $\sigma(c) = s$ . The operation  $\sigma$  keeps  $\pi$  within the NS or R sector.

The conformal weights and modular transformation matrices transform under  $\sigma$  as follows<sup>3</sup>:

$$\begin{aligned} h_{\sigma(\lambda)}^{N,K} &= h_\lambda^{N,K} + \frac{K(N-1)}{2N} - \frac{r_\lambda}{N}, & S_{\sigma(\lambda)\lambda'}^{N,K} &= e^{2\pi i r_{\lambda'}/N} S_{\lambda\lambda'}^{N,K} & \text{for } SU(N)_K \\ h_{\sigma(\pi)} &= h_\pi + \frac{1}{2}(\epsilon - 1) \bmod \mathbf{Z}, & S_{\sigma(\pi)\pi'} &= (-)^{\epsilon'} S_{\pi\pi'} & \text{for } SO(2N)_1 \end{aligned} \quad (2.15)$$

where  $\epsilon$  ( $\epsilon'$ ) is 0 or 1 according to whether  $\pi$  ( $\pi'$ ) is in the NS or R sector respectively.

If  $\Lambda$  obeys the selection rules (2.12), then  $J_1(\Lambda)$  and  $J_2(\Lambda)$  do so as well. Using eqs. (2.8) and (2.15), one may check that  $h_{J_1(\Lambda)} = h_{J_2(\Lambda)} = h_\Lambda$  modulo integers for  $\Lambda$  obeying eq. (2.12). The superconformal  $U(1)$  charge (2.10) is also invariant ( $\bmod 2\mathbf{Z}$ ) under  $J_1$  and  $J_2$ . Finally, from eq. (2.5), it follows that the modular transformation matrices are invariant:  $S_{J_1(\Lambda)\Lambda'} = S_{J_2(\Lambda)\Lambda'} = S_{\Lambda\Lambda'}$ . This implies that the determinant of  $S$  vanishes, since it has identical columns, and so cannot satisfy the modular group relation  $S^4 = 1$ .

To resolve this problem [4-6], one should identify  $J_1(\Lambda)$  and  $J_2(\Lambda)$  with  $\Lambda$ , rather than considering them to be distinct primary fields. The primary fields of the coset model are then equivalence classes of  $\Lambda$ , each of which contains  $mn(m+n)$  elements, except when  $m$ ,  $n$ , and  $k$  have a common divisor, in which case some of the equivalence classes – the

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<sup>3</sup> All expressions for the  $S$  matrices in ref. [16] should be complex conjugated to agree with those given in this paper. See the footnote on p. 868 of that reference.

“fixed points” – are smaller.<sup>4</sup> When  $m$ ,  $n$ , and  $k$  have no common divisor, the number of primary fields of  $G(m, n, k)$  is given by

$$N_{m,n,k} = 4 \binom{m+n+k-1}{k} \binom{m+n+k-1}{n+k} \binom{m+n+k-1}{m+k} \frac{m+n+k}{mn(m+n)}, \quad (2.16)$$

which is invariant under any permutation of  $m$ ,  $n$ , and  $k$ .<sup>5</sup> The “field identification” rescales the modular transformation matrix [7], giving rise to the factor  $mn(m+n)$  in eq. (2.5).

The connection between selection rules and field identification is most easily understood via the formalism of identification currents [7,8]. One first finds the identification currents of the coset model, which are simple currents with respect to which fields that obey the selection rules (2.12) have vanishing monodromy charge. Then one identifies any two fields related by an identification current. That is, primary fields of the coset model correspond to orbits of the identification currents. For  $G(m, n, k)$ , the identification currents are  $J_1(I)$  and  $J_2(I)$  [8], where  $I$  denotes the identity  $\begin{pmatrix} 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ .

The field identifications (2.14) allow us some freedom in choosing which set of representations  $\lambda_0$ ,  $\lambda_1$ ,  $\lambda_2$ ,  $\pi$ , and  $q$  to use to describe a given primary field. For example, we may use  $J_1$  to choose  $\lambda_1$  to be any element in its cominimal equivalence class. Then we can use  $J_2$  to choose  $\lambda_0$  to be any element in its class, without affecting the prior choice of  $\lambda_1$ . We will choose both  $\lambda_0$  and  $\lambda_1$  to be “cominimally reduced,” where a representation  $\lambda$  of  $SU(N)_K$  is said to be cominimally reduced if its tableau has fewer than  $K$  boxes in its first row, or equivalently, if  $a_0(\lambda) \neq 0$ . (Since  $\sum_{i=0}^{N-1} a_i(\lambda) = K$ , at least one of the extended Dynkin indices must be non-zero and so can be rotated into the zeroth position.)

Having fixed  $\lambda_0$  and  $\lambda_1$ , we still have some freedom to shift  $\lambda_2$ . Acting with  $J_2^{(m+n)x}$  will leave  $\lambda_0$  invariant for any integer  $x$ , since  $\sigma^{m+n}(\lambda_0) = \lambda_0$ . Since  $\sigma^{(m+n)x}(\lambda_2) = \sigma^{(m+n)x+nx'}(\lambda_2)$ , we can shift  $\lambda_2$  by  $(m, n)$  units along its orbit by choosing  $x$  to satisfy  $(m+n)x + nx' = (m, n)$ , that is,  $mx/(m, n) = 1 \bmod n/(m, n)$ , which is guaranteed to have a solution between 0 and  $n/(m, n)$ . This gives the identification

$$\begin{pmatrix} \lambda_0 & \pi \\ \lambda_1 & \lambda_2 & q \end{pmatrix} \approx \begin{pmatrix} \lambda_0 & \sigma^{m(m+n)x}(\pi) \\ \lambda_1 & \sigma^{(m,n)}(\lambda_2) & q - m(m+n)(m+n+k)x \end{pmatrix}. \quad (2.17)$$

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<sup>4</sup> We will deal with this situation in section 6.

<sup>5</sup> We thank M. Crescimanno for this observation.

With  $\lambda_0, \lambda_1$ , and  $\lambda_2$  fixed, there will generically be four possibilities for  $\pi$  and  $(m + n + k)$  allowed, inequivalent values of  $q$ , corresponding to  $4(m + n + k)$  distinct primary fields, as we will now show. If  $(m, n) = 1$ , the inequivalent values of  $a$ , defined in eq. (2.12), lie between 0 and  $n(m + n + k)$ , but the constraint (2.13) allows only  $(m + n + k)$  of these. If  $(m, n) \neq 1$ , then acting with  $J_1^{m(m+n)/(m,n)}$  takes  $\Lambda$  to  $\begin{pmatrix} \lambda_0 & \pi \\ \lambda_1 & \lambda_2 \\ & q + mn(m+n)(m+n+k)/(m,n) \end{pmatrix}$ , so we identify  $q$ 's differing by  $mn(m+n)(m+n+k)/(m,n)$ . There are then  $n(m+n+k)/(m,n)$  inequivalent values of  $a$ , but the constraint (2.13) reduces this by a factor of  $n/(m,n)$ , so again there are (generically)  $(m + n + k)$  allowed, inequivalent values of  $q$ . This will not be true, however, if there are short orbits.

### Short orbits of simple currents

When  $(N, K) \neq 1$ , it can happen that the fields  $\lambda, \sigma(\lambda), \dots, \sigma^{N-1}(\lambda)$  are not all distinct but that  $\sigma^{N/d}(\lambda) = \lambda$  for some divisor  $d$  of  $N$ . We then say that  $\lambda$  is in a “short orbit” of length  $N/d$ . The Dynkin indices of  $\lambda$  repeat in  $d$  groups of  $N/d$ :  $(a_0, a_1, \dots, a_{N/d-1}, a_0, a_1, \dots, a_{N/d-1}, \dots)$ , where  $\sum_{i=0}^{N/d-1} a_i = K/d$ , so in fact  $d$  must divide  $(N, K)$ .

If one or more of the representations  $\lambda_0, \lambda_1$ , or  $\lambda_2$  is in a short orbit, then there will be fewer than  $(m + n + k)$  inequivalent, allowed values of  $q$  for fixed  $\lambda_0, \lambda_1, \lambda_2$ , and  $\pi$ . For example, if  $\lambda_1$  is in a short orbit of length  $m/d$ , then acting with  $J_1^{m(m+n)/d}$  takes  $\Lambda$  to  $\begin{pmatrix} \lambda_0 & \sigma^{mn(m+n)/d}(\pi) \\ \lambda_1 & \lambda_2 \\ & q + mn(m+n)(m+n+k)/d \end{pmatrix}$ , forcing us to identify  $q$ 's that differ by  $mn(m+n)(m+n+k)/d$  (when  $mn(m+n)/d$  is even). There will then be  $4(m + n + k)/d$  distinct primary fields for fixed  $\lambda_0, \lambda_1$ , and  $\lambda_2$ . When  $mn(m+n)/d$  is odd, the identification interval for  $q$  is doubled, but in that case  $\pi$  and  $\sigma(\pi)$  do not correspond to distinct primary fields, so the number of primary fields remains  $4(m + n + k)/d$ .

### 3. Level-rank duality between $SU(N)_K$ and $SU(K)_N$

In this section, we briefly review the results of level-rank duality between the  $SU(N)_K$  and  $SU(K)_N$  WZW models. Level-rank duality denotes a correspondence, not necessarily an equivalence, between various quantities of the two models. First, it is a simple algebraic fact that the sum of central charges of dual theories obeys

$$c^{N,K} + c^{K,N} = NK - 1. \quad (3.1)$$

Second, if  $\lambda$  denotes an integrable representation of  $SU(N)_K$ , then  $\tilde{\lambda}$ , defined by exchanging the rows and columns of the Young tableau corresponding to  $\lambda$ , is an integrable representation of  $SU(K)_N$ . If  $\lambda$  is a cominimally-reduced representation, then the conformal weights of  $\lambda$  and  $\tilde{\lambda}$  satisfy [13,15]

$$h_{\lambda}^{N,K} + h_{\tilde{\lambda}}^{K,N} = \frac{r_{\lambda}}{2} \left(1 - \frac{r_{\lambda}}{NK}\right). \quad (3.2)$$

If  $\lambda$  is not cominimally reduced, then the sum differs from this by a known [16] integer. Similarly, the modular transformation matrices of dual theories are related by<sup>6</sup> [14-16]

$$S_{\lambda\lambda'}^{N,K} = \sqrt{\frac{K}{N}} e^{2\pi i r_{\lambda} r_{\lambda'}/NK} \left(S_{\tilde{\lambda}\tilde{\lambda}'}^{K,N}\right)^* \quad (3.3)$$

which by virtue of Verlinde's formula leads to an equality of fusion coefficients of  $SU(N)_K$  and  $SU(K)_N$  [14,16]

$$N_{\lambda\lambda'}^{\lambda''} = N_{\tilde{\lambda}\tilde{\lambda}'}^{\sigma^{\Delta}(\tilde{\lambda}'')}, \quad (3.4)$$

where  $\Delta = (r_{\lambda} + r_{\lambda'} - r_{\lambda''})/N \in \mathbf{Z}$ .

The number of primary fields of  $SU(N)_K$ , namely  $\binom{N+K-1}{K}$ , is not invariant under  $N \leftrightarrow K$ , hence the transpose map  $\lambda \rightarrow \tilde{\lambda}$  between primary fields of  $SU(N)_K$  and  $SU(K)_N$  cannot be one-to-one. Indeed, two cominimally equivalent representations will (often) transpose to the same dual representation. The transpose map, however, *is* one-to-one between cominimal equivalence classes, or simple current orbits. Moreover, the sizes of the orbits are correlated under this map: if  $\lambda$  is in a short orbit of length  $N/d$ , then  $\tilde{\lambda}$  is in a short orbit of length  $K/d$ . (We will demonstrate this in section 6.) Letting  $n_{\text{orbits}}$  be the number of orbits and  $d_i$  the divisor of the  $i^{\text{th}}$  orbit, we can write the number of primary fields of  $SU(N)_K$  and  $SU(K)_N$  as  $\sum_{i=1}^{n_{\text{orbits}}} (N/d_i)$  and  $\sum_{i=1}^{n_{\text{orbits}}} (K/d_i)$  respectively. The ratio of these is manifestly  $N/K$ , consistent with the expression above.

#### 4. Map between primary fields of $G(m, n, k)$ and $G(k, n, m)$

In this section, we provide strong evidence for the equivalence of the coset models

$$G(m, n, k) = \frac{\text{SU}(m+n)_k \times \text{SO}(2mn)_1}{\text{SU}(m)_{n+k} \times \text{SU}(n)_{m+k} \times \text{U}(1)_{mn(m+n)(m+n+k)}} \quad (4.1)$$

---

<sup>6</sup> See footnote 3.

and

$$G(k, n, m) = \frac{\mathrm{SU}(k+n)_m \times \mathrm{SO}(2kn)_1}{\mathrm{SU}(k)_{n+m} \times \mathrm{SU}(n)_{m+k} \times \mathrm{U}(1)_{kn(k+n)(m+n+k)}} \quad (4.2)$$

by establishing a one-to-one map between the primary fields of the theories, and showing the equivalence of their conformal weights (mod  $\mathbf{Z}$ ), their superconformal  $\mathrm{U}(1)$  charges (mod  $2\mathbf{Z}$ ), and their modular transformation matrices (and hence fusion rules).

The difference of the central charges of these two cosets

$$c^{m,n,k} - c^{k,n,m} = (c^{m+n,k} + c^{k,m+n}) - (c^{m,n+k} + c^{n+k,m}) + (mn - kn) = 0 \quad (4.3)$$

vanishes as a consequence of the level-rank relation (3.1). To specify a one-to-one map between the primary fields of  $G(m, n, k)$  and  $G(k, n, m)$ , we will define a map between the multi-indices

$$\Lambda = \begin{pmatrix} \lambda_0 & \pi \\ \lambda_1 & \lambda_2 \\ & q \end{pmatrix} \rightarrow \tilde{\Lambda} = \begin{pmatrix} \mu_0 & \rho \\ \mu_1 & \mu_2 \\ & \tilde{q} \end{pmatrix} \quad (4.4)$$

up to field identifications. The selection rules (2.12) for  $G(k, n, m)$  require

$$\begin{aligned} \tilde{q} &= -kr_{\mu_0} + (k+n)r_{\mu_1} + k(k+n)(\tilde{a} + \frac{1}{2}n\tilde{\epsilon}) \pmod{kn(k+n)(m+n+k)} \\ \tilde{q} &= nr_{\mu_0} - (k+n)r_{\mu_2} + n(k+n)(\tilde{b} + \frac{1}{2}k\tilde{\epsilon}) \pmod{kn(k+n)(m+n+k)} \end{aligned} \quad (4.5)$$

where  $\tilde{a}$  and  $\tilde{b}$  are integers, and  $\tilde{\epsilon} = 0$  or  $1$  for  $\rho \in \mathrm{NS}$  or  $\mathrm{R}$  respectively. Combining the two selection rules yields the constraint

$$k\tilde{a} - n\tilde{b} = r_{\mu_0} - r_{\mu_1} - r_{\mu_2} \pmod{kn(m+n+k)}, \quad (4.6)$$

which implies that  $r_{\mu_0} - r_{\mu_1} - r_{\mu_2}$  must be a multiple of  $(k, n)$ .

### Determining the map

Since the group factors  $\mathrm{SU}(k+n)_m$  and  $\mathrm{SU}(k)_{m+n}$  in  $G(k, n, m)$  are level-rank duals of factors in  $G(m, n, k)$ , the natural map between corresponding representations is Young tableau transposition [13-17,21]:

$$\begin{aligned} \mu_0 &= \tilde{\lambda}_1, \\ \mu_1 &= \tilde{\lambda}_0. \end{aligned} \quad (4.7)$$

That the level-rank duality map is only well-defined between cominimal equivalence classes dovetails with the fact that the primary fields of the coset theory are defined by  $\Lambda$  only up to field identification. To be precise about the map, we first use the field identifications (2.14) to ensure that both  $\lambda_0$  and  $\lambda_1$  are cominimally reduced.

Both  $\lambda_2$  and  $\mu_2$  are representations of  $SU(n)_{m+k}$ , but the naive guess

$$\mu_2 = \lambda_2 \quad (\text{incorrect}) \quad (4.8)$$

is incorrect. To understand why this is so, consider the chain of mappings

$$G(m, n, k) \rightarrow G(k, n, m) \rightarrow G(k, m, n) \rightarrow G(n, m, k). \quad (4.9)$$

Applying the maps (4.7) and (4.8) successively, we would arrive at

$$\begin{pmatrix} \lambda_0 & - \\ \lambda_1 & \lambda_2 & - \end{pmatrix} \rightarrow \begin{pmatrix} \lambda_0 & - \\ \lambda_2 & \lambda_1 & - \end{pmatrix} \quad (\text{incorrect}) \quad (4.10)$$

as the map from the fields of  $G(m, n, k)$  to  $G(n, m, k)$ . That this is wrong can most easily be seen by noting that the selection rules (2.12) are *not* invariant under eq. (4.10) together with  $m \leftrightarrow n$  because of a relative minus sign between them. By considering how the Dynkin diagrams of  $SU(m)$  and  $SU(n)$  are embedded in the Dynkin diagram of  $SU(m+n)$ , one realizes that the exchange of  $m$  and  $n$  must be accompanied by a  $\mathbf{Z}_2$  flip of each Dynkin diagram, which takes a representation  $\lambda \in SU(N)$  to its conjugate  $\bar{\lambda}$ , since  $a_i(\bar{\lambda}) = a_{N-i}(\lambda)$ . Thus, under  $m \leftrightarrow n$ , the correct mapping of representations is not eq. (4.10) but

$$\begin{pmatrix} \lambda_0 & - \\ \lambda_1 & \lambda_2 & - \end{pmatrix} \rightarrow \begin{pmatrix} \bar{\lambda}_0 & - \\ \bar{\lambda}_2 & \bar{\lambda}_1 & - \end{pmatrix} \quad (\text{correct}) \quad (4.11)$$

which preserves the selection rules (2.12), since  $r_{\bar{\lambda}} = -r_{\lambda} \bmod N$ .

One obvious way of obtaining eq. (4.11) from the chain of mappings (4.9) is to postulate

$$\mu_2 = \bar{\lambda}_2 \quad (\text{incorrect}). \quad (4.12)$$

This map, however, will not always be correct as it may violate the selection rule constraint (4.6). Since  $\lambda_0$  and  $\lambda_1$  are cominimally reduced, we have  $r_{\tilde{\lambda}_0} = r_{\lambda_0}$  and  $r_{\tilde{\lambda}_1} = r_{\lambda_1}$ , hence  $(r_{\mu_0} - r_{\mu_1} - r_{\mu_2}) = -(r_{\lambda_0} - r_{\lambda_1} - r_{\lambda_2}) \bmod n$ . While  $(r_{\lambda_0} - r_{\lambda_1} - r_{\lambda_2})$  is necessarily a multiple of  $(m, n)$  in order to satisfy eq. (2.13), it will not necessarily be a multiple of  $(k, n)$ , so the proposed map may not obey the constraint (4.6).

The correct map from  $\lambda_2$  to  $\mu_2$  is slightly more general:

$$\mu_2 = \sigma^v(\bar{\lambda}_2) \quad (\text{correct}) \quad (4.13)$$

with  $v$  an integer to be specified, not always zero. Since  $\sigma$  adds a row of width  $(m+k)$  to  $\bar{\lambda}_2$ , we have  $r_{\sigma^v(\bar{\lambda}_2)} = r_{\bar{\lambda}_2} + (m+k)v \bmod n$ , so eq. (4.6) implies

$$k\tilde{a} - n\tilde{b} = r_{\lambda_1} - r_{\lambda_0} - [-r_{\lambda_2} + (m+k)v] \bmod n, \quad (4.14)$$

which means that  $v$  must be chosen to satisfy

$$r_{\lambda_0} - r_{\lambda_1} - r_{\lambda_2} + (m+k)v = 0 \bmod (k, n). \quad (4.15)$$

Our specification of  $v$  below will automatically satisfy this constraint.

In defining the map from  $\pi \in \mathrm{SO}(2mn)_1$  to  $\rho \in \mathrm{SO}(2kn)_1$ , we assume that  $\tilde{\epsilon} = \epsilon$ , *i.e.*, the NS sector maps to the NS sector, and the R sector to the R sector. Given this, only a two-fold choice remains, namely whether  $\pi = 1$  maps to  $\rho = 1$  or to  $\rho = v$ , and similarly for other values of  $\pi$ . (Although the groups  $\mathrm{SO}(2mn)_1$  and  $\mathrm{SO}(2kn)_1$  differ, we will with slight abuse of notation use the same symbols for the conjugacy classes of each. Thus  $\rho = \pi$  means  $1 \in \mathrm{SO}(2mn)_1$  maps to  $1 \in \mathrm{SO}(2kn)_1$  and so forth.) We thus write this map

$$\rho = \sigma^u(\pi) \quad (4.16)$$

where the integer  $u$  is defined modulo 2. Thus, the map between  $\Lambda$  and  $\tilde{\Lambda}$  is

$$\begin{pmatrix} \lambda_0 & \pi \\ \lambda_1 & \lambda_2 & q \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{\lambda}_1 & \sigma^u(\pi) \\ \tilde{\lambda}_0 & \sigma^v(\bar{\lambda}_2) & \tilde{q} \end{pmatrix} \quad (4.17)$$

where  $v$ ,  $u$ , and  $\tilde{q}$  have yet to be specified.

As discussed in section 2, for fixed  $\lambda_0$ ,  $\lambda_1$ , and  $\lambda_2$ , there will generically be  $4(m+n+k)$  distinct primary fields of  $G(m, n, k)$ , labelled by  $\pi$  and  $q$ . Likewise, for fixed  $\mu_0$ ,  $\mu_1$ , and  $\mu_2$ ,  $G(k, n, m)$  will have  $4(m+n+k)$  primary fields, so it is reasonable to seek a one-to-one map. As we saw in the last paragraph of section 2, however, if one or more of  $\lambda_0$ ,  $\lambda_1$ , or  $\lambda_2$  belong to short orbits, the number of distinct primary fields in  $G(m, n, k)$  is fewer, appearing to endanger the one-to-one correspondence. In fact, the one-to-one correspondence is preserved: if for example  $\lambda_0$  is in a short orbit of length  $(m+n)/d$ , then  $\tilde{\lambda}_0$  is in a short orbit of length  $k/d$  (as will be shown in section 6), so the number of distinct primary fields of  $G(m, n, k)$  and  $G(k, n, m)$  is reduced by the same factor.

To further determine the map (4.17), we use the invariance of the superconformal  $\mathrm{U}(1)$  charges (2.10) of corresponding primary fields,  $Q_\Lambda = \tilde{Q}_{\tilde{\Lambda}}$ , which implies

$$f_{2kn}(\rho) - f_{2mn}(\pi) = \frac{\tilde{q} - q}{m+n+k} \bmod 2. \quad (4.18)$$

Eqs. (2.11) and (4.16) imply

$$f_{2kn}(\rho) - f_{2mn}(\pi) = u + \frac{1}{2}(k-m)n\epsilon \pmod{2}. \quad (4.19)$$

Likewise, the selection rules (2.12) and (4.5) yield

$$\frac{\tilde{q} - q}{m + n + k} = r_{\lambda_0} - r_{\lambda_1} + (k-m)(a + \frac{1}{2}n\epsilon) + k(k+n)\frac{(\tilde{a} - a)}{m + n + k} \pmod{\begin{cases} n, & n \text{ even}, \\ 2n, & n \text{ odd}. \end{cases}} \quad (4.20)$$

Combining eqs. (4.18), (4.19) and (4.20), we learn that  $k(k+n)(\tilde{a} - a)$  must be divisible by  $m + n + k$ , but since  $k(k+n)$  is not generically divisible by  $m + n + k$ , we will assume that  $\tilde{a} - a$  is, so that

$$\tilde{a} = a + (m + n + k)s \quad (4.21)$$

for some integer  $s$ . The relation (4.21) between  $a$  and  $\tilde{a}$  then determines  $u$ :

$$u = r_{\lambda_0} - r_{\lambda_1} + (k-m)a + k(k+n)s \pmod{2}. \quad (4.22)$$

To find  $s$ , we add the constraints (2.13) and (4.14) and use eq. (4.21) to find

$$(m + k)(a + ks + v) = 0 \pmod{n}. \quad (4.23)$$

If  $(m + k, n) = 1$ , it follows that  $a + ks + v$  must be divisible by  $n$ . We make the plausible assumption that this remains true even when  $(m + k, n) > 1$ :

$$a + ks + v = 0 \pmod{n} \quad (4.24)$$

which requires that

$$v = -a \pmod{(k, n)}. \quad (4.25)$$

Equation (4.25) will serve as our definition of  $v$ . To see that this satisfies the constraint (4.15), observe that eq. (4.24) together with eq. (2.13) implies

$$mv + mks + r_{\lambda_0} - r_{\lambda_1} - r_{\lambda_2} = 0 \pmod{n}, \quad (4.26)$$

hence eq. (4.15) follows. The mod  $(k, n)$  ambiguity in the definition (4.25) of  $v$  precisely corresponds to the ambiguity due to field identification, as we will see below. When  $(k, n) = 1$ , one is always free to choose  $v = 0$ .

Having chosen some  $v$  satisfying eq. (4.25), we set the right hand side of eq. (4.24) to  $tn$ , for some integer  $t$ , whence

$$-\frac{k}{(k,n)}s + \frac{n}{(k,n)}t = \frac{a+v}{(k,n)}. \quad (4.27)$$

Since  $k/(k,n)$  and  $n/(k,n)$  are relatively prime, this equation has a unique solution  $s$  modulo  $n/(k,n)$ . By eqs. (4.21) and (4.5),  $s$  determines  $\tilde{q}$  modulo  $kn(k+n)(m+n+k)/(k,n)$ , which from our discussion in section 2 is precisely the expected ambiguity due to field identification.

Given this solution  $s$  (chosen, say, to lie in the range  $0, \dots, \frac{n}{(k,n)} - 1$ ), we may now rewrite

$$\begin{aligned} a &= -ks + nt - v \\ \tilde{a} &= (m+n)s + nt - v \end{aligned} \quad (4.28)$$

and from eq. (4.22),

$$u = r_{\lambda_0} - r_{\lambda_1} + k(m+n)s + (k-m)(nt - v), \quad (4.29)$$

all in terms of  $s$ ,  $t$ , and  $v$ .

Let us now reconsider the mod  $(k,n)$  ambiguity in the definition of  $v$  in eq. (4.25). Suppose instead of  $v$  we chose

$$v' = v + (k,n). \quad (4.30)$$

Then using eqs. (4.27) and (4.22), we have

$$s' = s - \tilde{x}, \quad u' = u - k(k+n)\tilde{x} \quad (4.31)$$

where  $k\tilde{x}/(k,n) = 1 \pmod{n/(k,n)}$ . With this choice of  $v$ , the map from  $G(m,n,k)$  to  $G(k,n,m)$  becomes

$$\begin{pmatrix} \lambda_0 & \pi \\ \lambda_1 & \lambda_2 & q \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{\lambda}_1 & \sigma^{u-k(k+n)\tilde{x}}(\pi) \\ \tilde{\lambda}_0 & \sigma^{v+(k,n)}(\bar{\lambda}_2) & \tilde{q} - k(k+n)(m+n+k)\tilde{x} \end{pmatrix}. \quad (4.32)$$

The difference between eqs. (4.17) and (4.32), however, is just the field equivalence (2.17).

## Equivalence of conformal weights and modular transformation matrices

Having determined the map between primary fields of  $G(m, n, k)$  and  $G(k, n, m)$ , we now proceed to establish the equivalence of their conformal weights and modular transformation matrices. The conformal weights (2.8) of  $\Lambda \in G(m, n, k)$  and  $\tilde{\Lambda} \in G(k, n, m)$  differ by

$$\begin{aligned} h_{\Lambda}^{m,n,k} - h_{\tilde{\Lambda}}^{k,n,m} &= (h_{\lambda_0}^{m+n,k} + h_{\tilde{\lambda}_0}^{k,m+n}) - (h_{\lambda_1}^{m,n+k} + h_{\tilde{\lambda}_1}^{n+k,m}) \\ &\quad + (h_{\sigma^v(\tilde{\lambda}_2)}^{n,m+k} - h_{\lambda_2}^{n,m+k}) + (h_{\pi}^{2mn} - h_{\sigma^u(\pi)}^{2kn}) + (h_{\tilde{q}} - h_q) \quad \text{mod } \mathbf{Z}. \end{aligned} \quad (4.33)$$

The first two pieces of this are given by the level-rank relation (3.2),

$$\begin{aligned} h_{\lambda_0}^{m+n,k} + h_{\tilde{\lambda}_0}^{k,m+n} &= \frac{r_{\lambda_0}}{2} \left[ 1 - \frac{r_{\lambda_0}}{k(m+n)} \right], \\ h_{\lambda_1}^{m,n+k} + h_{\tilde{\lambda}_1}^{n+k,m} &= \frac{r_{\lambda_1}}{2} \left[ 1 - \frac{r_{\lambda_1}}{m(n+k)} \right]. \end{aligned} \quad (4.34)$$

We also need [16]

$$h_{\sigma^v(\tilde{\lambda}_2)}^{n,m+k} = h_{\sigma^{-v}(\lambda_2)}^{n,m+k} = h_{\lambda_2}^{n,m+k} + \frac{(m+k)v(n-v)}{2n} + \frac{vr_{\lambda_2}}{n} \quad \text{mod } \mathbf{Z} \quad (4.35)$$

together with (2.9)

$$h_{\pi}^{2mn} - h_{\sigma^u(\pi)}^{2kn} = \begin{cases} \frac{1}{2}u \quad \text{mod } \mathbf{Z}, & \pi \in \text{NS} \\ \frac{1}{8}n(m-k), & \pi \in \text{R}. \end{cases} \quad (4.36)$$

The difference of U(1) weights is

$$\begin{aligned} h_{\tilde{q}} - h_q &= \frac{\tilde{q}^2}{2kn(k+n)(m+n+k)} - \frac{q^2}{2mn(m+n)(m+n+k)} \\ &= \frac{r_{\lambda_0}^2}{2k(m+n)} - \frac{r_{\lambda_1}^2}{2m(k+n)} - \frac{v}{n} [r_{\lambda_0} - r_{\lambda_1} + \frac{1}{2}(m-k)v + mks] + \frac{1}{2}\epsilon(r_{\lambda_0} - r_{\lambda_1}) \\ &\quad + \frac{1}{2}\epsilon(m-k)v + \frac{1}{2}n(k-m)(t + \frac{1}{2}\epsilon)^2 + \frac{1}{2}k(m+n)s(s+\epsilon) \quad \text{mod } \mathbf{Z} \end{aligned} \quad (4.37)$$

via eqs. (2.12), (4.5), and (4.28). Putting all these together, and using eq. (4.26), it is straightforward to see that  $h_{\Lambda}^{m,n,k} - h_{\tilde{\Lambda}}^{k,n,m} = 0 \text{ mod } \mathbf{Z}$  in the R ( $\epsilon = 1$ ) sector. In the NS ( $\epsilon = 0$ ) sector,

$$h_{\Lambda}^{m,n,k} - h_{\tilde{\Lambda}}^{k,n,m} = \frac{1}{2} [r_{\lambda_0} - r_{\lambda_1} + (m-k)v + n(k-m)t^2 + k(m+n)s^2 + u] \quad \text{mod } \mathbf{Z}. \quad (4.38)$$

Using the expression (4.29) for  $u$ , this reduces to

$$h_{\Lambda}^{m,n,k} - h_{\tilde{\Lambda}}^{k,n,m} = \frac{1}{2} [n(k-m)t(t+1) + k(m+n)s(s+1)] \pmod{\mathbf{Z}} \quad (4.39)$$

which vanishes mod  $\mathbf{Z}$ . Thus we have shown that in general

$$h_{\Lambda}^{m,n,k} = h_{\tilde{\Lambda}}^{k,n,m} \pmod{\mathbf{Z}}. \quad (4.40)$$

Next, consider the ratio of modular transformation matrices (2.5) for  $G(m, n, k)$  and  $G(k, n, m)$ :

$$\frac{S_{\Lambda\Lambda'}^{m,n,k}}{S_{\tilde{\Lambda}\tilde{\Lambda}'}^{k,n,m}} = \frac{m(m+n)}{k(k+n)} \frac{S_{\lambda_0\lambda'_0}}{S_{\tilde{\lambda}_0\tilde{\lambda}'_0}} \frac{S_{\lambda_1\lambda'_1}^*}{S_{\tilde{\lambda}_1\tilde{\lambda}'_1}} \frac{S_{\lambda_2\lambda'_2}^*}{S_{\sigma^v(\bar{\lambda}_2)\sigma^{v'}(\bar{\lambda}'_2)}^*} \frac{S_{\pi\pi'}^*}{S_{\sigma^u(\pi)\sigma^{u'}(\pi')}^*} \frac{S_{qq'}^*}{S_{\tilde{q}\tilde{q}'}^*}. \quad (4.41)$$

Again we need the level-rank relations (3.3)

$$\begin{aligned} \frac{S_{\lambda_0\lambda'_0}}{S_{\tilde{\lambda}_0\tilde{\lambda}'_0}^*} &= \sqrt{\frac{k}{m+n}} \exp\left[\frac{2\pi i r_{\lambda_0} r_{\lambda'_0}}{k(m+n)}\right], \\ \frac{S_{\lambda_1\lambda'_1}^*}{S_{\tilde{\lambda}_1\tilde{\lambda}'_1}} &= \sqrt{\frac{n+k}{m}} \exp\left[-\frac{2\pi i r_{\lambda_1} r_{\lambda'_1}}{m(n+k)}\right], \end{aligned} \quad (4.42)$$

together with<sup>7</sup> [16]

$$\frac{S_{\lambda_2\lambda'_2}^*}{S_{\sigma^v(\bar{\lambda}_2)\sigma^{v'}(\bar{\lambda}'_2)}^*} = \exp\left(\frac{2\pi i}{n} [(m+k)vv' - vr_{\lambda'_2} - v'r_{\lambda_2}]\right) \quad (4.43)$$

and (2.6)

$$\frac{S_{\pi\pi'}}{S_{\sigma^u(\pi)\sigma^{u'}(\pi')}^*} = \begin{cases} 1 & \pi, \pi' \in \text{NS}, \\ (-1)^u & \pi \in \text{NS}, \pi' \in \text{R}, \\ (-1)^{u'} & \pi \in \text{R}, \pi' \in \text{NS}, \\ (-1)^{u+u'} i^{n(k-m)} & \pi, \pi' \in \text{R}. \end{cases} \quad (4.44)$$

The ratio of the U(1) modular transformation matrices

$$\begin{aligned} \frac{S_{qq'}^*}{S_{\tilde{q}\tilde{q}'}^*} &= \sqrt{\frac{k(k+n)}{m(m+n)}} \exp\left(\frac{2\pi i}{m+n+k} \left[ \frac{qq'}{mn(m+n)} - \frac{\tilde{q}\tilde{q}'}{kn(k+n)} \right]\right) \\ &= \sqrt{\frac{k(k+n)}{m(m+n)}} \exp\left(2\pi i \left[ -\frac{r_{\lambda_0} r_{\mu_0}}{k(m+n)} + \frac{r_{\lambda_1} r_{\mu_1}}{m(k+n)} + \frac{vr_{\lambda'_2} + v'r_{\lambda_2}}{n} \right. \right. \\ &\quad \left. \left. - \frac{(k+m)vv'}{n} - \frac{\epsilon u' + \epsilon' u}{2} + \frac{\epsilon \epsilon' n(m-k)}{4} \right]\right) \end{aligned} \quad (4.45)$$

is obtained using eqs. (4.26) and (4.29). The factors in the product (4.41) exactly cancel, establishing the equivalence of the modular transformation matrices

$$S_{\Lambda\Lambda'}^{m,n,k} = S_{\tilde{\Lambda}\tilde{\Lambda}'}^{k,n,m}. \quad (4.46)$$

The equivalence of the fusion rules then automatically follows from Verlinde's formula.

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<sup>7</sup> See footnote 3.

## 5. Map between the chiral rings of $G(m, 1, k)$ and $G(k, 1, m)$

In this section, we describe the map between the chiral rings of  $G(m, n, k)$  and  $G(k, n, m)$  when  $n = 1$ .<sup>8</sup> The chiral ring is composed of chiral primary fields, those that saturate the bound  $h_\Lambda \geq \frac{1}{2} |Q_\Lambda|$ . For  $G(m, 1, k)$  these have a simple characterization [5]

$$\Lambda^{\text{chiral}} = \begin{pmatrix} \lambda_0 & 1 \\ \lambda_1 & -q \end{pmatrix} \quad (5.1)$$

where, if the Dynkin indices of  $\lambda_0 \in \text{SU}(m+1)_k$  are  $(a_1, \dots, a_m)$ , then the Dynkin indices of  $\lambda_1 \in \text{SU}(m)_{k+1}$  are  $(a_1, \dots, a_{m-1})$ , and  $q = r_{\lambda_0}$ . The chiral primaries are in one-to-one correspondence with the primary fields of  $\text{SU}(m+1)_k$ , which number  $\binom{m+k}{k}$ , and their conformal weights and superconformal  $U(1)$  charges are proportional to the number of boxes of the corresponding tableau,  $h_\Lambda = -\frac{1}{2}Q_\Lambda = r_{\lambda_0}/[2(m+1+k)] \bmod \mathbf{Z}$ . In fact the Poincaré polynomial [6] for  $G(m, 1, k)$  just counts the number of  $\text{SU}(m+1)_k$  representations graded by the number of boxes in their tableaux.

The number of chiral primaries is manifestly invariant under  $m \leftrightarrow k$ . What is the relation between the chiral primary (5.1) of  $G(m, 1, k)$  and the corresponding chiral primary

$$\tilde{\Lambda}^{\text{chiral}} = \begin{pmatrix} \mu_0 & 1 \\ \mu_1 & -\tilde{q} \end{pmatrix} \quad (5.2)$$

of  $G(k, 1, m)$ ? The answer is that  $\mu_0 \in \text{SU}(k+1)_m$  is given by  $\tilde{\lambda}_0$ , the tableau transpose of  $\lambda_0 \in \text{SU}(m+1)_k$ , and  $\tilde{q} = q$ , since tableau transposition preserves the number of boxes. Note that tableau transposition generates a one-to-one map between the primary fields of  $\text{SU}(m+1)_k$  and  $\text{SU}(k+1)_m$ .<sup>9</sup> This is in contrast to the usual level-rank duality between  $\text{SU}(N)_K$  and  $\text{SU}(K)_N$  in which tableau transposition only generates a correspondence between cominimal equivalence classes.

At first sight, this map between chiral primaries seems different from the map between primary fields prescribed in the previous section, in which  $\mu_0 = \tilde{\lambda}_1$  and  $\mu_1 = \tilde{\lambda}_0$ , but we will show that the two maps are equivalent. First we consider the case in which  $\lambda_0$ , and therefore  $\lambda_1$ , are cominimally reduced. Then, by the prescription given in section 4, the chiral primary (5.1) maps to

$$\tilde{\Lambda}^{\text{chiral}} = \begin{pmatrix} \tilde{\lambda}_1 & \sigma^{ka_m}(1) \\ \tilde{\lambda}_0 & -q + k(m+1+k)a_m \end{pmatrix}. \quad (5.3)$$

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<sup>8</sup> In this case, the Kazama-Suzuki model is a Landau-Ginzburg theory, but not necessarily otherwise [6].

<sup>9</sup> See conjecture 1 of ref. [26].

where  $a_m$  is the last Dynkin index of  $\lambda_0$ . By acting  $k(k + 1 - a_m)$  times with  $J_1$  on this field, we obtain

$$\tilde{\Lambda}^{\text{chiral}} = \begin{pmatrix} \sigma^{a_m}(\tilde{\lambda}_1) & 1 \\ \tilde{\lambda}_0 & -q \end{pmatrix} \quad (5.4)$$

but since the tableau for  $\lambda_1$  is the same as the tableau for  $\lambda_0$  with  $a_m$  columns of  $m$  boxes prepended to it, it follows that  $\sigma^{a_m}(\tilde{\lambda}_1)$  is simply  $\tilde{\lambda}_0$ , so the field (5.4) is just eq. (5.2).

Next suppose  $\lambda_0$  is not cominimally reduced, but has  $\tilde{a}_k$  rows of boxes of width  $k$  at the top. To implement the map to  $G(k, 1, m)$ , we first need to cominimally reduce  $\lambda_0$  by acting  $m\tilde{a}_k$  times with  $J_1$ , which gives

$$\Lambda^{\text{chiral}} = \begin{pmatrix} \sigma^{-\tilde{a}_k}(\lambda_0) & \sigma^{m\tilde{a}_k}(1) \\ \lambda_1 & -q + m(m + 1 + k)\tilde{a}_k \end{pmatrix}. \quad (5.5)$$

Then we map this to  $G(k, 1, m)$ , obtaining

$$\tilde{\Lambda}^{\text{chiral}} = \begin{pmatrix} \tilde{\lambda}_1 & \sigma^{k\tilde{a}_m}(1) \\ \sigma^{-\tilde{a}_k}(\lambda_0) & -q + k(m + 1 + k)a_m \end{pmatrix}. \quad (5.6)$$

Finally, we act  $k(k + 1 - a_m)$  times with  $J_1$  to obtain

$$\tilde{\Lambda}^{\text{chiral}} = \begin{pmatrix} \sigma^{a_m}(\tilde{\lambda}_1) & 1 \\ \sigma^{-\tilde{a}_k}(\lambda_0) & -q \end{pmatrix}. \quad (5.7)$$

Since  $\sigma^{a_m}(\tilde{\lambda}_1)$  is just  $\tilde{\lambda}_0$ , and  $\sigma^{-\tilde{a}_k}(\lambda_0)$  is just  $\tilde{\lambda}_0$  with  $\tilde{a}_k$  columns of height  $k$  removed, this is precisely the field (5.2). Thus, the map between primary fields defined in section 4 is equivalent to the map between chiral primaries described below eq. (5.2).

## 6. Fixed point resolution

If  $m, n$ , and  $k$  have a common divisor  $p > 1$ , then some of the equivalence classes of fields  $\Lambda = \begin{pmatrix} \lambda_0 & \pi \\ \lambda_1 & \lambda_2 & q \end{pmatrix}$  have fewer than  $mn(m + n)$  elements. These are called fixed-point fields, and are those for which  $\lambda_0, \lambda_1$ , and  $\lambda_2$  all belong to short orbits [8]:

$$\sigma^{\hat{m}+\hat{n}}(\lambda_0) = \lambda_0, \quad \sigma^{\hat{m}}(\lambda_1) = \lambda_1, \quad \sigma^{\hat{n}}(\lambda_2) = \lambda_2, \quad (6.1)$$

where  $\hat{m} = m/p$ ,  $\hat{n} = n/p$ , and  $\hat{k} = k/p$ . Each fixed-point equivalence class  $\Lambda$  actually corresponds to a set of  $p$  distinct primary fields, distinguished by the index  $i$ ,

$$\Lambda_i^{\text{FP}} = \begin{pmatrix} \lambda_0 & \pi & \\ \lambda_1 & \lambda_2 & q; & i \end{pmatrix}, \quad i = 1, \dots, p. \quad (6.2)$$

Due to this resolution of fixed points, the characters and modular transformation matrices are modified [7] from the naive prescriptions (2.4) and (2.5). Each row and column of the modular matrix  $S$  corresponding to a fixed-point field is resolved into  $p$  rows and columns, with

$$S_{\Lambda_i \Lambda'_{i'}}^{\text{resolved}} = \frac{1}{p^2} S_{\Lambda \Lambda'}^{m,n,k} E_{ii'} + \Gamma_{\Lambda_i \Lambda'_{i'}}^{m,n,k} \quad (6.3)$$

for some  $\Gamma_{\Lambda_i \Lambda'_{i'}}$ , where  $E_{ii'}$  is 1 for any  $i$  and  $i'$ . The conformal weights of the fields  $\Lambda_i$  are independent of  $i$ , so the modular matrix  $T$  is not modified.

Schellekens and Yankielowicz [7] show that the modular group relations  $(ST)^3 = S^2$  and  $S^4 = 1$  obeyed by the resolved modular transformation matrices imply that  $\Gamma$  and  $T$ , restricted to the fixed-point fields, satisfy the same relations. This suggests that they may be identified (up to a 12<sup>th</sup> root of unity, which preserves the modular group relations) as the modular transformation matrices  $\hat{S}$  and  $\hat{T}$  of an auxiliary “fixed-point” theory [27],

$$\begin{aligned} \Gamma_{\Lambda_i \Lambda'_{i'}}^{m,n,k} &= e^{-i\pi w/2} \hat{S}_{\hat{\Lambda} \hat{\Lambda}'}^{m,n,k} P_{ii'} \\ T_{\Lambda_i \Lambda_i} &= e^{i\pi w/6} \hat{T}_{\hat{\Lambda} \hat{\Lambda}'} \end{aligned} \quad (6.4)$$

where  $P_{ii'} = \delta_{ii'} - (1/p)E_{ii'}$ ,  $w$  is some integer, and  $\hat{\Lambda}$  denotes the “projection” of the fixed-point field  $\Lambda$  onto a field in the fixed-point theory.

For  $p$  prime, Schellekens [8] has shown that the fixed-point theory corresponding to  $G(m, n, k)$  is

$$\hat{G}(m, n, k) = \frac{\text{SU}(\hat{m} + \hat{n})_{\hat{k}} \times \text{SO}(2mn)_1}{\text{SU}(\hat{m})_{\hat{n} + \hat{k}} \times \text{SU}(\hat{n})_{\hat{m} + \hat{k}} \times \text{U}(1)_{\hat{m}\hat{n}(\hat{m} + \hat{n})(\hat{m} + \hat{n} + \hat{k})}}. \quad (6.5)$$

Observe that  $\hat{G}(m, n, k)$  differs from  $G(\hat{m}, \hat{n}, \hat{k})$  in that the orthogonal group factor is  $\text{SO}(2mn)_1$ , not  $\text{SO}(2\hat{m}\hat{n})_1$ . Given this fixed-point theory, we need to determine the projection  $\Lambda \rightarrow \hat{\Lambda}$ . Recall that the Dynkin indices of a representation  $\lambda$  of  $\text{SU}(N)_K$  in a short orbit repeat in groups of  $\hat{N}$ ,  $(a_0, a_1, \dots, a_{\hat{N}-1}, a_0, a_1, \dots, a_{\hat{N}-1}, \dots)$ , since  $\sigma^{\hat{N}}(\lambda) = \lambda$ . Here  $\hat{N} = N/p$  and  $\hat{K} = K/p$ . We associate [28] with  $\lambda$  a representation  $\hat{\lambda}$  of  $\text{SU}(\hat{N})_{\hat{K}}$  with Dynkin indices  $(a_0, a_1, \dots, a_{\hat{N}-1})$ . One may then show that

$$r_\lambda = pr_{\hat{\lambda}} + \frac{1}{2}p(p-1)\hat{N}\hat{K} \quad (6.6)$$

and [28]

$$\left( h_\lambda^{N,K} - \frac{c^{N,K}}{24} \right) = \left( h_{\hat{\lambda}}^{\hat{N},\hat{K}} - \frac{c^{\hat{N},\hat{K}}}{24} \right) + \frac{NK - \hat{N}\hat{K}}{24}. \quad (6.7)$$

Since  $\lambda_0$  and  $\lambda_1$  are in short orbits,  $q$  is divisible by  $p^2$ , by eqs. (2.12) and (6.6). Moreover,  $J_1^{m(m+n)/p^2}$  acting on a fixed-point field allows us to identify  $q$ 's differing by  $mn(m+n)(m+n+k)/p^2$ . Hence, we may rescale  $q$  to  $\hat{q} = q/p^2$ , and regard  $\hat{q}$  as labeling a representation of  $U(1)_{\hat{m}\hat{n}(\hat{m}+\hat{n})(\hat{m}+\hat{n}+\hat{k})}$ . In short,  $\Lambda$  is projected onto the multi-index

$$\hat{\Lambda} = \begin{pmatrix} \hat{\lambda}_0 & \pi \\ \hat{\lambda}_1 & \hat{\lambda}_2 \\ & \hat{q} \end{pmatrix} \quad (6.8)$$

belonging to  $\hat{G}(m, n, k)$ .

Using eqs. (2.8) and (6.7), we calculate the difference between the conformal weights of  $\Lambda \in G(m, n, k)$  and  $\hat{\Lambda} \in \hat{G}(m, n, k)$

$$\left( h_{\Lambda}^{m,n,k} - \frac{c^{m,n,k}}{24} \right) = \left( \hat{h}_{\hat{\Lambda}}^{m,n,k} - \frac{\hat{c}^{m,n,k}}{24} \right) + \frac{\hat{m}\hat{n}(1-p^2)}{12} \pmod{\mathbf{Z}} \quad (6.9)$$

where  $\hat{h}_{\hat{\Lambda}}^{m,n,k}$  and  $\hat{c}^{m,n,k}$  are the conformal weights and central charge of the fixed-point theory. This implies via eq. (6.4) that  $w = \hat{m}\hat{n}(1-p^2)$  and thus

$$\Gamma_{\Lambda_i \Lambda'_i}^{m,n,k} = \exp\left(2\pi i \frac{\hat{m}\hat{n}(p^2-1)}{4}\right) \hat{S}_{\hat{\Lambda}\hat{\Lambda}'}^{m,n,k} P_{ii'} \quad \text{for } p \text{ prime.} \quad (6.10)$$

Although no other resolution of the fixed points is known, no proof exists that this solution is unique.

### Field identification in the fixed-point theory

The identifications (2.14) of fields in  $G(m, n, k)$  induces an identification of fields in  $\hat{G}(m, n, k)$

$$\begin{aligned} J_1(\hat{\Lambda}) &= \begin{pmatrix} \sigma(\hat{\lambda}_0) & \sigma^n(\pi) \\ \sigma(\hat{\lambda}_1) & \hat{\lambda}_2 \\ & \hat{q} + \hat{n}(\hat{m} + \hat{n} + \hat{k}) \end{pmatrix}, \\ J_2(\hat{\Lambda}) &= \begin{pmatrix} \sigma(\hat{\lambda}_0) & \sigma^m(\pi) \\ \hat{\lambda}_1 & \sigma(\hat{\lambda}_2) \\ & \hat{q} - \hat{m}(\hat{m} + \hat{n} + \hat{k}) \end{pmatrix}. \end{aligned} \quad (6.11)$$

Observe that  $\sigma^n$ , not  $\sigma^{\hat{n}}$ , acts on  $\pi \in SO(2mn)_1$ . To fully define the action of  $J_1$  and  $J_2$  on the resolved fixed-point fields  $\Lambda_i^{\text{FP}}$ , we need to specify further that

$$\begin{aligned} J_1(i) &= \begin{cases} i, & \text{prime } p > 2 \\ \sigma^{\hat{n}}(i), & p = 2 \end{cases} \\ J_2(i) &= \begin{cases} i, & \text{prime } p > 2 \\ \sigma^{\hat{m}}(i), & p = 2 \end{cases} \end{aligned} \quad (6.12)$$

where  $\sigma(1) = 2$  and  $\sigma(2) = 1$ . Since

$$\begin{aligned} P_{\sigma(i)i'} &= -P_{ii'} & \text{for } p = 2 \\ E_{\sigma(i)i'} &= E_{ii'} \end{aligned} \tag{6.13}$$

the assignment (6.12) guarantees via eq. (6.6) that  $\Gamma_{J_1(\Lambda_i)\Lambda'_i}^{m,n,k} = \Gamma_{J_2(\Lambda_i)\Lambda'_i}^{m,n,k} = \Gamma_{\Lambda_i\Lambda'_i}^{m,n,k}$ , so that the resolved modular transformation matrix (6.3) remains invariant under  $J_1$  and  $J_2$ , as required for field identification.

In the case we are considering, where  $m$ ,  $n$ , and  $k$  have a prime greatest common divisor  $p$ , the number of primary fields in  $G(m, n, k)$  is given by

$$\begin{aligned} N_{m,n,k} &= 4 \left[ \binom{m+n+k-1}{k} \binom{m+n+k-1}{n+k} \binom{m+n+k-1}{m+k} \right. \\ &\quad \left. - \binom{\hat{m}+\hat{n}+\hat{k}-1}{\hat{k}} \binom{\hat{m}+\hat{n}+\hat{k}-1}{\hat{n}+\hat{k}} \binom{\hat{m}+\hat{n}+\hat{k}-1}{\hat{m}+\hat{k}} \right] \frac{m+n+k}{mn(m+n)} \\ &\quad + 4p \binom{\hat{m}+\hat{n}+\hat{k}-1}{\hat{k}} \binom{\hat{m}+\hat{n}+\hat{k}-1}{\hat{n}+\hat{k}} \binom{\hat{m}+\hat{n}+\hat{k}-1}{\hat{m}+\hat{k}} \frac{\hat{m}+\hat{n}+\hat{k}}{\hat{m}\hat{n}(\hat{m}+\hat{n})} \\ &= \frac{4(m+n+k)!(m+n+k-1)!^2}{m!n!k!(m+n)!(m+k)!(n+k)!} + \left( p - \frac{1}{p^2} \right) \frac{4(\hat{m}+\hat{n}+\hat{k})!(\hat{m}+\hat{n}+\hat{k}-1)!^2}{\hat{m}\hat{n}\hat{k}!(\hat{m}+\hat{n})!(\hat{m}+\hat{k})!(\hat{n}+\hat{k})!} \end{aligned} \tag{6.14}$$

where the factor of  $p$  in the third line is the multiplicity of the resolved fixed-point fields. The expression (6.14) is manifestly invariant under  $k \leftrightarrow m$ , so a one-to-one map between resolved primary fields of  $G(m, n, k)$  and  $G(k, n, m)$  is still possible.

### Map between fixed-point fields

Under level-rank duality, short orbits of  $SU(N)_K$  of length  $N/p$  are mapped onto short orbits of  $SU(K)_N$  of length  $K/p$ . The proof of this is simple: let  $\lambda$  belong to an orbit of length  $N/p$ , and no shorter than  $N/p$ . Project it onto  $\hat{\lambda}$ , a representation of  $SU(\hat{N})_{\hat{K}}$ . The transpose of  $\hat{\lambda}$  is  $\tilde{\hat{\lambda}}$ , a representation of  $SU(\hat{K})_{\hat{N}}$ . But  $\tilde{\hat{\lambda}}$  is equal to  $\hat{\tilde{\lambda}}$ , the projection of  $\tilde{\lambda}$  onto  $SU(\hat{K})_{\hat{N}}$ . Therefore  $\tilde{\lambda}$  belongs to an orbit of  $SU(K)_N$  no longer than  $K/p$ . Reversing the argument guarantees that it also belongs to an orbit no shorter than  $K/p$ .

For a fixed-point field  $\Lambda$ , the representations  $\lambda_0$ ,  $\lambda_1$ , and  $\lambda_2$  are all in short orbits (6.1). The argument of the previous paragraph implies  $\sigma^{\hat{k}}(\tilde{\lambda}_0) = \tilde{\lambda}_0$  and  $\sigma^{\hat{k}+\hat{n}}(\tilde{\lambda}_1) = \tilde{\lambda}_1$ . It is also true that  $\sigma^{\hat{n}}(\sigma^v(\tilde{\lambda}_2)) = \sigma^v(\tilde{\lambda}_2)$ , so  $\tilde{\Lambda}$  is a fixed-point field of  $G(k, n, m)$ . Thus, resolved fixed points of  $G(m, n, k)$  are mapped to resolved fixed points of  $G(k, n, m)$ ,  $\Lambda_i^{\text{FP}} \rightarrow \tilde{\Lambda}_{\tau(i)}^{\text{FP}}$ , where  $\tau(i)$  has not yet been specified.

The resolved modular transformation matrix for fixed points of  $G(k, n, m)$  is

$$S_{\tilde{\Lambda}_i \tilde{\Lambda}'_{i'}}^{\text{resolved}} = \frac{1}{p^2} S_{\tilde{\Lambda} \tilde{\Lambda}'}^{k, n, m} E_{ii'} + \Gamma_{\tilde{\Lambda}_i \tilde{\Lambda}'_{i'}}^{k, n, m} \quad (6.15)$$

with

$$\Gamma_{\tilde{\Lambda}_i \tilde{\Lambda}'_{i'}}^{k, n, m} = \exp \left( 2\pi i \frac{\hat{k}\hat{n}(p^2 - 1)}{4} \right) \hat{S}_{\tilde{\Lambda} \tilde{\Lambda}'}^{k, n, m} P_{ii'} \quad \text{for } p \text{ prime,} \quad (6.16)$$

where  $\hat{S}_{\tilde{\Lambda} \tilde{\Lambda}'}^{k, n, m}$  is the modular transformation matrix of the fixed-point theory  $\hat{G}(k, n, m)$ .

Using eq. (4.17), the field  $\tilde{\Lambda}$  is projected onto

$$\hat{\tilde{\Lambda}} = \begin{pmatrix} \hat{\tilde{\lambda}}_1 & & \sigma^u(\pi) \\ \hat{\tilde{\lambda}}_0 & \widehat{\sigma^v(\tilde{\lambda}_2)} & \hat{\tilde{q}} \end{pmatrix} \quad (6.17)$$

with  $\hat{\tilde{q}} = \tilde{q}/p^2$ . Equation (6.17) may also be written

$$\hat{\tilde{\Lambda}} = \begin{pmatrix} \hat{\tilde{\lambda}}_1 & & \sigma^u(\pi) \\ \hat{\tilde{\lambda}}_0 & \sigma^v(\hat{\tilde{\lambda}}_2) & \hat{\tilde{q}} \end{pmatrix} \quad (6.18)$$

so the map between fields of the fixed-point theories  $\hat{G}(m, n, k)$  and  $\hat{G}(k, n, m)$  is

$$\begin{pmatrix} \hat{\tilde{\lambda}}_0 & \pi \\ \hat{\tilde{\lambda}}_1 & \hat{\tilde{\lambda}}_2 \\ & \hat{\tilde{q}} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{\tilde{\lambda}}_1 & & \sigma^u(\pi) \\ \hat{\tilde{\lambda}}_0 & \sigma^v(\hat{\tilde{\lambda}}_2) & \hat{\tilde{q}} \end{pmatrix}. \quad (6.19)$$

### Equivalence of resolved modular transformation matrices

We now show the equivalence of the resolved modular transformation matrices (6.3) and (6.15). The equivalence of  $S_{\Lambda \Lambda'}^{m, n, k} E_{ii'}$  and  $S_{\tilde{\Lambda} \tilde{\Lambda}'}^{k, n, m} E_{\tau(i)\tau(i')}$  was previously established in section 4, independent of  $\tau(i)$ . Consider the ratio

$$\frac{\Gamma_{\Lambda_i \Lambda'_{i'}}^{m, n, k}}{\Gamma_{\tilde{\Lambda}_{\tau(i)} \tilde{\Lambda}'_{\tau(i')}}^{k, n, m}} = e^{\pi i \hat{n}(\hat{m} - \hat{k})(p^2 - 1)/2} \frac{\hat{m}(\hat{m} + \hat{n})}{\hat{k}(\hat{k} + \hat{n})} \frac{S_{\hat{\lambda}_0 \hat{\lambda}'_0}}{S_{\tilde{\lambda}_0 \tilde{\lambda}'_0}^*} \frac{S_{\hat{\lambda}_1 \hat{\lambda}'_1}^*}{S_{\tilde{\lambda}_1 \tilde{\lambda}'_1}^*} \frac{S_{\hat{\lambda}_2 \hat{\lambda}'_2}^*}{S_{\sigma^v(\hat{\lambda}_2) \sigma^{v'}(\hat{\lambda}'_2)}^*} \frac{S_{\pi \pi'}}{S_{\sigma^u(\pi) \sigma^{u'}(\pi')}} \frac{S_{\hat{q} \hat{q}'}^*}{S_{\tilde{q} \tilde{q}'}^*} \frac{P_{ii'}}{P_{\tau(i)\tau(i')}} \quad (6.20)$$

where

$$\begin{aligned} \frac{S_{\hat{\lambda}_0 \hat{\lambda}'_0}}{S_{\tilde{\lambda}_0 \tilde{\lambda}'_0}^*} &= \sqrt{\frac{\hat{k}}{\hat{m} + \hat{n}}} \exp \left[ \frac{2\pi i r_{\hat{\lambda}_0} r_{\hat{\lambda}'_0}}{\hat{k}(\hat{m} + \hat{n})} \right], \\ \frac{S_{\hat{\lambda}_1 \hat{\lambda}'_1}^*}{S_{\tilde{\lambda}_1 \tilde{\lambda}'_1}^*} &= \sqrt{\frac{\hat{n} + \hat{k}}{\hat{m}}} \exp \left[ - \frac{2\pi i r_{\hat{\lambda}_1} r_{\hat{\lambda}'_1}}{\hat{m}(\hat{n} + \hat{k})} \right], \\ \frac{S_{\hat{\lambda}_2 \hat{\lambda}'_2}^*}{S_{\sigma^v(\hat{\lambda}_2) \sigma^{v'}(\hat{\lambda}'_2)}^*} &= \exp \left( \frac{2\pi i}{\hat{n}} \left[ (\hat{m} + \hat{k})vv' - vr_{\hat{\lambda}'_2} - v'r_{\hat{\lambda}_2} \right] \right), \end{aligned} \quad (6.21)$$

and

$$\frac{S_{\hat{q}\hat{q}'}^*}{S_{\hat{\tilde{q}}\hat{q}'}^*} = \sqrt{\frac{\hat{k}(\hat{k} + \hat{n})}{\hat{m}(\hat{m} + \hat{n})}} \exp\left(\frac{2\pi i}{\hat{m} + \hat{n} + \hat{k}} \left[ \frac{\hat{q}\hat{q}'}{\hat{m}\hat{n}(\hat{m} + \hat{n})} - \frac{\hat{\tilde{q}}\hat{\tilde{q}}'}{\hat{k}\hat{n}(\hat{k} + \hat{n})} \right] \right) = \frac{S_{qq'}^*}{S_{\tilde{q}\tilde{q}'}^*}. \quad (6.22)$$

The easiest way to evaluate eq. (6.20) is to take its ratio with eq. (4.41) and use eq. (6.6) to find

$$\frac{\Gamma_{\Lambda_i \Lambda'_{i'}}^{m,n,k}}{\Gamma_{\tilde{\Lambda}_{\tau(i)} \tilde{\Lambda}'_{\tau(i')}}^{k,n,m}} = \exp\left(\pi i(p-1) \left[ r_{\lambda_0} - r_{\lambda_1} - (\hat{m} + \hat{k})v + r_{\lambda'_0} - r_{\lambda'_1} - (\hat{m} + \hat{k})v' \right] \right) \frac{P_{ii'}}{P_{\tau(i)\tau(i')}} \quad (6.23)$$

for  $p$  prime. If we define

$$\tau(i) = \begin{cases} i, & \text{prime } p > 2 \\ \sigma^{r_{\lambda_0} - r_{\lambda_1} - (\hat{m} + \hat{k})v}(i), & p = 2 \end{cases} \quad (6.24)$$

and use eq. (6.13), the ratio (6.23) becomes unity, and the equivalence of the resolved modular transformation matrices (6.3) and (6.15) is established. Consequently, the fusion rules between the resolved primary fields of  $G(m, n, k)$  and  $G(k, n, m)$  are identical.

## 7. Concluding remarks

We have provided strong evidence that the Kazama-Suzuki models  $G(m, n, k)$  and  $G(k, n, m)$  are isomorphic by constructing a one-to-one map between their primary fields and demonstrating the equivalence of corresponding conformal weights, superconformal  $U(1)$  charges, modular transformation matrices, and fusion rules. We have shown that the equivalence continues to hold when  $m$ ,  $n$ , and  $k$  possess a common prime divisor  $p$  and the theories contain fixed points. (We expect the equivalence to hold for any  $m$ ,  $n$ , and  $k$ , but the proof of this would require knowledge of the fixed-point theory when their greatest common divisor is not prime.) Primary fields corresponding to resolved fixed points in  $G(m, n, k)$  are mapped onto resolved fixed points of  $G(k, n, m)$ . This is simpler than in other superconformal coset models [21] in which resolved fixed-point fields are mapped to nonfixed points and vice versa, and thus in which fixed-point resolution is indispensable for the map. There is, however, some subtlety in the map between the resolved fixed points (6.19) and (6.24) when  $p = 2$ .

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